Thermal state pointer can enormously improve the precision of weak measurement

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In most studies on post-selected weak measurement, the zero-mean Gaussian state (i.e., pure state) is initialized as a pointer state. Recently, several theoretical studies give the limits that using fisher information metric, weak measurement amplification strategy can not outperform the standard metrology strategy. However, we show that the limits, especially for the imaginary part of weak value, is no longer tenable if thermal state is initialized as a pointer state. By adjusting the temperature, we can improve accuracy of the measurement precision.

This manuscript is only a simple calculation proof for "thermal state pointer can improve precision of post-selected weak measurement" and a detailed illustration for the meaning of this work and other related work will be given in the future.

I. WEAK MEASUREMENT AMPLIFICATION MODEL

Let us begin with a brief review of postselected weak measurement using thermal state pointer [1]. In the standard scenario of weak measurement [2], the interaction Hamiltonian between the system and the pointer is

$$\hat{H} = \chi(t) A \otimes q, \tag{1}$$

where A is a system observable, q is the position observable of the pointer and $\chi(t)$ is a narrow pulse function with interaction strength χ. Suppose the initial system state is $|\psi_i\rangle = \sum_{i=1}^{2} c_i|a_i\rangle$ (a two-dimensional) with $A|a_i\rangle = a_i|a_i\rangle$, where $c_1 = \cos\frac{\theta_i}{2}$, $c_2 = e^{i\varphi}\sin\frac{\theta_i}{2}$. Then we consider the initial pointer state as $\rho_{th}(z) = (1-z)\sum_{n=0} z^n|n\rangle_m\langle n|_m$ with $z = e^{-\hbar\omega_m/k_B T}$ (a continuous degree of freedom), where k_B is the Boltzmann constant and T is the temperature.

The joint state of the total system can be easily initially prepared

$$|\psi_i\rangle\langle\psi_i| \otimes \rho_{th}(z) = (1-z)\sum_{i,j=1,n=0}^{2} c_i c_j^* z^n |a_i\rangle\langle a_j| \otimes |n\rangle_m\langle n|_m \tag{2}$$

where c_i is a normalised amplitude.

The time evolution operator is

$$U = e^{-i\chi A q}. \tag{3}$$

After the interaction (3), the time evolution of the total system is given by

$$\rho(z) = (1-z)\sum_{i,j=1,n=0}^{2} c_i c_j^* z^n |a_i\rangle\langle a_j| \otimes e^{-i\chi a_i q} |n\rangle_m\langle n|_m e^{i\chi a_j q} \tag{4}$$

For Eq. (4), changing to p-representation in rectangular coordinate, we can obtain

$$\rho(z,p) = (1-z)\sum_{i,j=1,n=0} c_i c_j^* z^n |a_i\rangle\langle a_j| \int\int_{-\infty}^{\infty} \frac{1}{2^n n!}(-i)^n i^n H_n(\sqrt{2}\sigma p_i) H_n(\sqrt{2}\sigma p_j') \phi_0(p_i)\phi_0(p_j^{'})\mathrm{d}p\mathrm{d}p^{'}|p\rangle\langle p^{'}| \tag{5}$$

where H_n is Hermite Polynomial and $\phi_0(p_i) = (2\pi\frac{1}{4\sigma^2})^{-1/4}\exp(-\sigma^2 p_i^2)$ with $p_i = p + a_i\chi$ and σ is zero-point fluctuation.

Under this dynamics, each of the eigenstates $|a_i\rangle$ of the system observable A is entangled with the pointer state wavefunctions, which is translated by the different $a_i\chi$ proportional to the eigenvalue a_i. When $|a_i - a_j|\chi$ is much larger than the width ($\sqrt{\frac{1+z}{1-z}}\frac{1}{2\sigma}$) of $\rho_{th}(z)$, this becomes a strong measurement, meaning that the overlap $\mathcal{O}_{ij} := (\frac{1+z}{1-z})^{-1/2}\sum_{i\neq j}\int\exp[\frac{4\sigma^2 p_i p_j z - 2\sigma^2(p_i^2+p_j^2)z^2}{1-z^2}]\phi_0(p_i)\phi_0(p_j)\mathrm{d}p$ between each pair of shifted wavefunctions is vanishingly small. So the pointer states corresponding to different eigenvalues become completely separated. However, when $|a_i - a_j|\chi$ is relatively small and the wavefunctions are no longer well resolved, the measurement is said to be weak [3].

When the postselected state of the measured system $|\psi_f\rangle = \sum_{i=1}^{2} c_i'|a_i\rangle$ with $c_1' = \cos\frac{\theta_f}{2}$ and $c_2' = \sin\frac{\theta_f}{2}$ is performed for the total system (5), the reduced state of the pointer becomes

$$\rho'(z,p) = (1-z)\sum_{i,j=1,n=0} c_i c_i'^* c_j^* c_j' z^n \int\int_{-\infty}^{\infty}\frac{1}{2^n n!} H_n(\sqrt{2}\sigma p_i) H_n(\sqrt{2}\sigma p_j')\phi_0(p_i)\phi_0(p_j^{'})\mathrm{d}p\mathrm{d}p^{'}|p\rangle\langle p^{'}|. \tag{6}$$

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After measuring the particle in the p basis, and using identity

$$\psi(p)=\int_{-\infty}^{\infty}\delta(p-p')\psi(p')\mathrm{d}p' \tag{7}$$

and Mehler's Hermite Polynomial Formula

$$\sum_{n=0}^{\infty}\frac{H_n(x)H_n(y)}{n!}(\frac{1}{2}w)^n \tag{8}$$
$$=(1-w^2)^{-1/2}\exp[\frac{2xyw-(x^2+y^2}{1-w^2})],$$

the probability distribution of (6) over p becomes

$$\begin{aligned}P_{\mathrm{wma}}(p)=&\frac{1}{A_m}(2\pi\frac{1+z}{1-z}\frac{1}{4\sigma^2})^{-1/2}[r^2\\&\exp[-\frac{(p+a_1\chi)^2}{2\frac{1+z}{1-z}\frac{1}{4\sigma^2}}]+rt(e^{i\varphi}+e^{-i\varphi})\\&\exp[-\frac{(p+\frac{a_2-za_1}{1-z})^2}{4\frac{1+z}{1-z}\frac{1}{4\sigma^2}}-\frac{(p+\frac{a_1-za_2}{1-z})^2}{4\frac{1+z}{1-z}\frac{1}{4\sigma^2}}]\\&+t^2\exp[-\frac{(p+a_2\chi)^2}{2\frac{1+z}{1-z}\frac{1}{4\sigma^2}}],\end{aligned} \tag{9}$$

where $r=\cos\frac{\theta_i}{2}\cos\frac{\theta_f}{2}$, $t=\sin\frac{\theta_i}{2}\sin\frac{\theta_f}{2}$, and

$$A_m=r^2+t^2+2rt\exp[-\frac{(a_1-a_2)^2}{2}\frac{1+z}{1-z}\chi^2\sigma^2]\cos\varphi \tag{10}$$

represents the probability of successful postselection. The analyses above apply equally if q and p are interchanged, so that the interaction induces the displacement in momentum space.

When the postselected state of the measured system $|f\rangle=\sum_{i=1}^{2}c_i'|a_i\rangle$ with $c_1'=\cos\frac{\theta_f}{2}$ and $c_2'=\sin\frac{\theta_f}{2}$ is performed for the total system (4), the reduced state of the pointer becomes

$$\begin{aligned}\rho'(z)=(1-z)&\sum_{i,j=1,n=0}c_ic_i^{*\prime}c_j^*c_j'z^ne^{-i\chi a_iq}\\&|n\rangle_m\langle n|_m\,e^{i\chi a_jq}\end{aligned} \tag{11}$$

For Eq. (11), changing to q-representation in rectangular coordinate, we can obtain

$$\begin{aligned}\rho'(z,q)=(1-z)&\sum_{i,j=1,n=0}c_ic_i^{*\prime}c_j^*c_j'z^n\int\int_{-\infty}^{\infty}\\&\frac{1}{2^nn!}e^{-i\chi a_iq}H_n(\frac{q}{\sqrt{2}\sigma})H_n(\frac{q'}{\sqrt{2}\sigma})\\&\phi_0(q)\phi_0(q')e^{i\chi a_jq'}\mathrm{d}q\mathrm{d}q'|q\rangle\langle q'|\end{aligned} \tag{12}$$

where $\phi_0(q)=(2\pi\sigma^2)^{-1/4}\exp(-\frac{q^2}{4\sigma^2})$.

After measuring the particle in the q basis, the probability distribution of (11) over q becomes

$$\begin{aligned}\tilde{P}_{\mathrm{wma}}(q)=&\frac{1}{A_m}(2\pi\frac{1+z}{1-z}\sigma^2)^{-1/2}\exp[-\frac{q^2}{2\frac{1+z}{1-z}\sigma^2}]\\&[r^2+rt(e^{i\varphi}\exp[i(a_2-a_1)\chi q]\\&+e^{-i\varphi}\exp[-i(a_2-a_1)\chi q])+t^2].\end{aligned} \tag{13}$$

II. METRIC

In the precision metrology, a parameter estimation interested can be captured by the Fisher information [4]. The Fisher information is a functional on such conditional probability distributions, and is defined:

$$F_\chi[P(s|\chi)]=\int_{-\infty}^{\infty}\frac{(\partial_\chi P(s|\chi))^2}{P(s|\chi)}\mathrm{d}s\,. \tag{14}$$

The sensitive estimate of an unknown parameter χ is given by the observed statistics [5], i.e., Cramér-Rao bound limits

$$\mathrm{Var}(\chi)\geq\frac{1}{NF_\chi} \tag{15}$$

where N is the number of independent trials.

If the postselected weak measurement is applied to the precision metrology of a parameter estimation, the whole process is called weak measurement amplification strategy (i.e., WMA strategy). Moreover, we define a *standard* strategy without the postselection. It refers to the benchmark measurement strategy completely ignoring degree of freedom of the system. For Eq. (5), one traces over the degree of freedom of the system and measures the particle in the p basis to give

$$P_{\mathrm{std}}(p,z)=\sum_i|c_i|^2|\phi(p+a_i\chi)|^2\,, \tag{16}$$

where $\phi(p+a_i\chi)=(2\pi\frac{1+z}{1-z}\frac{1}{4\sigma^2})^{-1/2}\exp[-(p+a_i\chi)^2/(2\frac{1+z}{1-z}\frac{1}{4\sigma^2})]$.

The weighted sum (16) is a convex combination: by controlling the c_i one can mix the probability densities corresponding to the different eigenvalues of A. An immediate optimization for the standard strategy presents itself: The Fisher information metric is convex [6], meaning that any such mixing of probability distributions will be sub-optimal. One should therefore choose the c_i to filter the probability distribution with eigenvalue a_* that gives the highest Fisher information. Now

$$\begin{aligned}F_\chi[P_{\mathrm{std}}(p,z)]&=F_\chi[|\phi(p+a_*\chi)|^2]\\&=|a_*|^2F_p[|\phi(p)|^2]\\&=|a_*|^2(\frac{1+z}{1-z}\frac{1}{4\sigma^2})^{-1}.\end{aligned} \tag{17}$$

However, when $z=0$, the expression of (17) becomes

$$\begin{aligned}F_\chi[P_{\mathrm{std}}(p,z=0)]&=|a_*|^2F_p[|\phi_0(p)|^2]\\&=|a_*|^2(\frac{1}{4\sigma^2})^{-1},\end{aligned} \tag{18}$$

which shows the highest Fisher information of the pure Gaussian state in a *standard* strategy. It can be seen from (15) that in a *standard* strategy using pure Gaussian state give higher estimate of an unknown parameter χ than thermal state. Therefore, using pure Gaussian state pointer in the paper can be seen as the benchmark in the *standard* strategy.

However, our interesting attention in this paper is the Cramér-Rao bound of the WMA strategy, and how it compares to that of the standard strategy. In the limit of $N \to \infty$, their ratio is equal to $A_m F_\chi[P_{\rm wma}(p)]$ and $A_m F_\chi[P_{\rm wma}(q)]$ to $F_\chi[P_{\rm std}(p, z=0)]$, respectively. The formers are corrected by the probability of successful postselection.

III. IDEAL DETECTOR

Here we consider a stable and ideal detector (i.e., without technical imperfections).

In the WMA strategy, if choosing to measure in momentum space or in position space, one will obtain the displacement proportional to real part of the weak value or one proportional to imaginary part of the weak value, respectively. The corresponding conditional probability distribution is given by (9) or (13). Then taking a ratio of the WMA strategy to the standard strategy, for (9) and (18) we give

$$\begin{aligned}
&\frac{A_m F_\chi[P_{\rm wma}(p)]}{F_\chi[P_{\rm std}(p, z=0)]} \\
&= A_m \int_{-\infty}^{\infty} \frac{(\partial_\chi P_{\rm wma}(p))^2}{P_{\rm wma}(p)} \mathrm{d}p / [|a_*|^2 (\frac{1}{4\sigma^2})^{-1}]
\end{aligned} \tag{19}$$

if $\varphi = 0$. The numerator of (19) is the quantum Fisher information for the momentum's displacement proportional to real weak values. Note that $A_m F_\chi[P_{\rm wma}(p)]$ is no larger than the quantum Fisher information of the joint system state (5) without postselection [7], i.e., $F_{\rm total} = (ra_1 - ta_2)^2 (\frac{1+z}{1-z}\frac{1}{4\sigma^2})^{-1}$. However, in weak measurement limit, i.e., $\chi\sigma \to 0$, $A_m \int_{-\infty}^{\infty} \frac{(\partial_\chi P_{\rm wma}(p))^2}{P_{\rm wma}(p)} \mathrm{d}p$ is equal to $(ra_1 - ta_2)^2 (\frac{1+z}{1-z}\frac{1}{4\sigma^2})^{-1}$. Therefore, $\frac{A_m F_\chi[P_{\rm wma}(p)]}{F_\chi[P_{\rm std}(p,z=0)]} \leq \frac{1-z}{1+z}$, and the equality holds up if and only if $\theta_i = \frac{\pi}{4}$ and $\theta_f = -\frac{\pi}{4}$. It is obvious that quantum Fisher information for real weak values using thermal state pointer can be no advantage for the purpose of estimating χ.

However for (13) and (18) we give

$$\begin{aligned}
\frac{A_m F_\chi[\tilde{P}_{\rm wma}(q)]}{F_\chi[P_{\rm std}(p, z=0)]} &= \frac{(a_1 - a_2)^2}{2} [\frac{1+z}{1-z}\sigma^2 \\
&- \exp(-k^2/2)(k^2 \frac{1+z}{1-z}\sigma^2 - \frac{1+z}{1-z}\sigma^2) \cos\varphi \\
&- (\frac{\exp(-k^2/2) k^2 \frac{1+z}{1-z}\sigma^2 \cos^2\varphi}{\exp(k^2/2) - \cos\varphi})] / [|a_*|^2 (\frac{1}{4\sigma^2})^{-1}] \\
&= \frac{(a_1 - a_2)^2}{8|a_*|^2} \frac{1+z}{1-z} [1 + \exp(-k^2/2)(1-k^2)\cos\varphi \\
&- \frac{\exp(-k^2) k^2 \cos^2\varphi}{1 - \exp(-k^2/2)\cos\varphi}]
\end{aligned} \tag{20}$$

if $\theta_i = \frac{\pi}{4}$ and $\theta_f = -\frac{\pi}{4}$, where $k^2 = (a_1 - a_2)^2 \chi^2 \frac{1+z}{1-z}\sigma^2$. The numerator of (20) is the quantum Fisher information for the position's displacement proportional to imaginary weak values.

Suppose the eigenvalues $a_1 = 1$, $a_2 = -1$, implying that $|a_*| = 1$. In weak measurement regime, i.e., $k \ll 1$ and $\varphi \ll 1$, for example, $k^2 = 0.0005$ with $z = 0$, $\varphi = 0.05$, $\frac{A_m F_\chi[\tilde{P}_{\rm wma}(q)]}{F_\chi[P_{\rm std}(p,z=0)]} = 0.8327$ and $k^2 = 0.0095$ with $z = 0.9$, $\varphi = 0.05$, $\frac{A_m F_\chi[\tilde{P}_{\rm wma}(q)]}{F_\chi[P_{\rm std}(p,z=0)]} = 3.9478$. Moreover, when $\varphi = \pi/2$, $\frac{A_m F_\chi[\tilde{P}_{\rm wma}(q)]}{F_\chi[P_{\rm std}(p,z=0)]} = \frac{1+z}{1-z}/2$. These results indicate that as z grows, the ratio of (20) can exceed 1. In other words, by adjusting the temperature T, we can give better estimate of an unknown parameter χ. Therefore, the result breaks the inequality constraint in [8–10]

$$P_{\rm postselection} F_{\rm weak\ value} \leq F_{\rm standard}, \tag{21}$$

where $P_{\rm postselection}$ is the postselection success probability. It is obvious that post-selected weak measurement using thermal state pointer, corresponding to the displacement proportional to imaginary weak values, can increase the measurement precision.

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